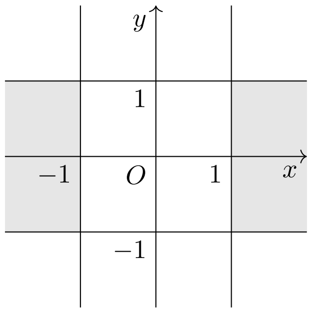
1101. Factorising, $x^6(3x^4+8)=0$. The latter factor has no real roots, as its value is always at least 8. So, the solution is $x=0$.

1102. Comparing coefficients of x^2, we require $a=3$. This gives the coefficients of x as $12+2b=2$, so $b=-5$. Then, for the constant term, we need $3-5+c=-3$, so $c=-1$. This gives

$$3(2x+1)^2-5(2x+1)-1.$$

1103. The region is



1104. In a large population, dependence in selections is negligible. Hence, the probability that any datum lies between the quartiles is $\frac{1}{2}$. So, the probability that all ten do is

$$p=\left(\tfrac{1}{2}\right)^{10}=\tfrac{1}{1024}.$$

1105. Multiplying up by both denominators:

$$\begin{aligned}&(2x-1)(5x-4)=(3x-2)(4x-3)\\ \Longrightarrow\ &10x^2-13x+4=12x^2-17x+6\\ \Longrightarrow\ &2x^2-4x+2=0\\ \Longrightarrow\ &x=1.\end{aligned}$$

1106. Substituting $x=a$, the y values are $y=\mathrm{f}(a)$ and $y=\mathrm{g}(a)=\mathrm{f}'(a)(a-a)+\mathrm{f}(a)=\mathrm{f}(a)$. Hence, the curves intersect at $x=a$. Also, the gradient of the line $y=\mathrm{g}(x)$ is $\mathrm{f}'(a)$, which is the same as that of $y=\mathrm{f}(x)$ at $x=a$. Hence, the graphs are tangent at $x=a$.

— Nota Bene —

This is the formula for a generic tangent line to $y=\mathrm{f}(x)$ at $x=a$:

$$y=\mathrm{f}'(a)(x-a)+\mathrm{f}(a).$$

1107. (a) By Pythagoras, the half-base has length 8 cm. So, using similar triangles, the half-width w of the shaded rectangle is given by

$$\frac{6}{h}=\frac{8}{8-w}.$$

Solving this gives $w=8-\frac{4h}{3}$, so the area is

$$A=2wh=16h-\frac{8h^2}{3}.$$

(b) Setting the derivative $\frac{dA}{dh}$ to zero gives

$$\begin{aligned}&16-\frac{16h}{3}=0\\ \Longrightarrow\ &h=3.\end{aligned}$$

Substituting $h=3$ gives $A=24$ cm^2, which is half the area of the triangle, as required.

— Nota Bene —

The area function is a negative quadratic, which guarantees that our stationary value is indeed a maximum, as opposed to a minimum.

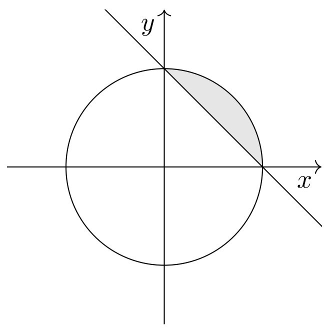
1108. (a) Redistributing factors of 10,

$$\begin{aligned}&2.3\times10^n+1.2\times10^{n-1}\\ \equiv\ &2.3\times10^n+0.12\times10^n\\ \equiv\ &2.42\times10^n.\end{aligned}$$

(b) Again redistributing,

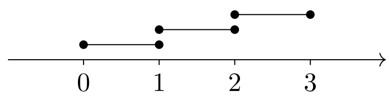
$$\begin{aligned}&5\times10^n+9.7\times10^{n+1}\\ \equiv\ &0.5\times10^{n+1}+9.7\times10^{n+1}\\ \equiv\ &10.2\times10^{n+1}\\ \equiv\ &1.02\times10^{n+2}.\end{aligned}$$

1109. Sketching, we require the area of the region below:



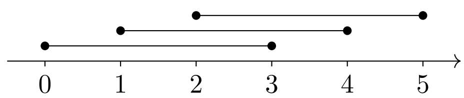
The area of a quarter-circle sector is $\pi/4$, and the area of the subtracted triangle is $1/2$, which gives the area of the shaded segment as $A=\pi/4-1/2$.

1110. (a) The intervals are



So, the union is $[0,3]$.

(b) The intervals are



So, the intersection is $[2,3]$.

1111. Both can exist, as trivial cases:

(a) $a = 0$, $b = 1$, $c = 1$ gives a linear function.

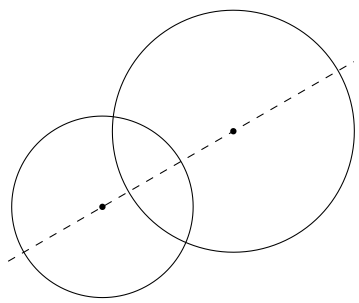
(b) $m = 1$, $n = 3$ gives a prime product.

1112. The output transformation scales the area by k, but the input transformation scales it by $\frac{1}{k}$, so the overall scale factor is 1.

1113. Solving simultaneously or by inspection, the pair intersect at $(1, 1)$. The radii to this point have vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{j}$, which are perpendicular. Hence, since tangent and radius are perpendicular, the circles are normal.

— Nota Bene —

The circles are also normal at the second point of intersection, although the question doesn't require this. Indeed, if two circles are normal at one point of intersection, then they *must*, by symmetry along the line of centres (dashed below), be normal at the other point of intersection.



1114. Since $\mathrm{f}''(x)$ has degree 4, $\mathrm{f}(x)$ must have degree 6. So $n = 5$. Differentiating twice,

$$\begin{aligned} & \mathrm{f}(x) = x^6 + x^5 \\ \Longrightarrow\ & \mathrm{f}'(x) = 6x^5 + 5x^4 \\ \Longrightarrow\ & \mathrm{f}''(x) = 30x^4 + 20x^3 \\ & \qquad = 10x^3(3x + 2). \end{aligned}$$

Hence $a = 10$, $b = 3$ and $c = 2$.

1115. Assume, for a contradiction, that a pentagon has five interior angles, each of which is smaller than 108°. Then the sum of the interior angles must be less than $5 \times 108° = 540°$. But the interior angles of a pentagon add to 540°. This is a contradiction. So, at least one angle must satisfy $\theta \geq 108°$. □

1116. (a) Using $s = ut + \frac{1}{2}at^2$ for the first gap,

$$\begin{aligned} & 50 = 2.2u + \tfrac{1}{2}a \cdot 2.2^2 \\ \Longrightarrow\ & 50 = 2.2u + 2.42a. \end{aligned}$$

Considering both gaps together,

$$\begin{aligned} & 100 = 4u + \tfrac{1}{2}a \cdot 4^2 \\ \Longrightarrow\ & 100 = 4u + 8a. \end{aligned}$$

(b) Solving simultaneously,

$$\begin{aligned} a &= \tfrac{250}{99} = 2.53 \text{ ms}^{-2} \text{ (3sf)} \\ u &= \tfrac{1975}{99} = 19.9 \text{ ms}^{-1} \text{ (3sf)}. \end{aligned}$$

1117. We assume the domain is $\mathbb{R}$, so the range of the cosine function is $[-1, 1]$. The range of $(\cos x + 3)$ is then $[2, 4]$, so the range of g is $[8, 64]$.

1118. (a) $H_1 : \rho < 0$, since the researcher is looking for *negative* correlation.

(b) The given critical value is $r_c = -0.264$. Since $-0.426 < -0.264$, the test statistic r is more extreme than the critical value r_c, so the result is significant. The sample provides sufficient evidence to reject H_0. There does appear to be negative correlation in the population.

1119. To enact this transformation, we use two variable replacements:

- replace x by $x - a$ for translation by $a\mathbf{i}$,
- replace y by $y - b$ for translation by $b\mathbf{j}$.

This gives the new graph as $x - a = (y - b)^2$.

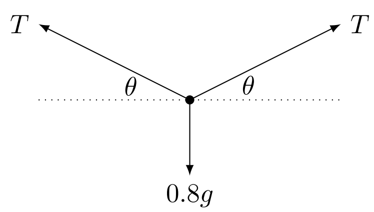
1120. (a) We require $x - 1 \geq 0$, so the broadest possible real domain is $[1, \infty)$.

(b) We require $1 - x \geq 0$, so the broadest possible real domain is $(-\infty, 1]$.

1121. The question is "Determine...", so full explanation is required, not just an answer. The formula gives

$$\begin{aligned} x &= \frac{-\sqrt{24} \pm \sqrt{24 + 4\sqrt{2} \cdot \sqrt{8}}}{2\sqrt{2}} \\ &= \frac{-\sqrt{24} \pm \sqrt{40}}{2\sqrt{2}} \\ &= \frac{-\sqrt{48} \pm \sqrt{80}}{4} \\ &= -\sqrt{3} \pm \sqrt{5}. \end{aligned}$$

1122. Using the lengths, the angle between each section of line and the horizontal is $\theta = \arctan \frac{0.2}{2}$. The force diagram for the bag is



Resolving vertically gives $2T \sin\theta - 0.8g = 0$, so the tension is 39.4 N (3sf).

1123. Since AB is a diameter, the centre O is at $(3, 3)$. Then, since CD is a diameter, $\overrightarrow{CO} = \overrightarrow{OD}$. This gives the coordinates of D as $\left(6, 3 - \sqrt{8}\right)$.

1124. The mean of the interior angles of a quadrilateral is 90°. Since, in this case, the interior angles are in AP, they are distributed symmetrically around their mean, so the difference between the largest and 90° is the same as the difference between the smallest and 90°. The smallest must be greater than 0°, so the largest must be less than 180°. □

1125. Expanding the brackets and splitting the fraction,

$$\begin{aligned}&40\int_1^8 x^{\frac{5}{3}} + 2x^{\frac{2}{3}} + x^{-\frac{1}{3}}\,dx \\ &= 40\left[\tfrac{3}{8}x^{\frac{8}{3}} + \tfrac{6}{5}x^{\frac{5}{3}} + \tfrac{3}{2}x^{\frac{2}{3}}\right]_1^8 \\ &= 5616 - 123 \\ &= 5493, \text{ as required.}\end{aligned}$$

1126. We know that $\text{f}''(x) = a$, for some constant a. Integrating twice gives $\text{f}(x) = px^2 + qx + r$, for some constants p, q, r (where $p = \frac{1}{2}a$). This is all quadratic and linear functions.

1127. Since there are 400 gradians in a full circle, arc length and sector area should be given as fractions out of 400. So, the formulae, for θ in gradians, are

$$l = \frac{\theta}{400}2\pi r, \qquad A = \frac{\theta}{400}\pi r^2.$$

1128. Using $|a| = |b| \iff a^2 = b^2$,

$$\begin{aligned}&(x^2 - x)^2 = (x-1)^2 \\ \Longrightarrow\ &(x^2 - x)^2 - (x-1)^2 = 0 \\ \Longrightarrow\ &x^2(x-1)^2 - (x-1)^2 = 0 \\ \Longrightarrow\ &(x-1)^2(x^2-1) = 0 \\ \Longrightarrow\ &(x-1)^3(x+1) = 0 \\ \Longrightarrow\ &x = \pm 1.\end{aligned}$$

1129. The expression is a difference of two squares:

$$\begin{aligned}&\left(x - a + \sqrt{b}\right)\left(x - a - \sqrt{b}\right) \\ \equiv\ &(x-a)^2 - \left(\sqrt{b}\right)^2 \\ \equiv\ &x^2 - 2ax + a^2 - b.\end{aligned}$$

Since a and b are integers, so are $-2a$ and $a^2 - b$. Hence, the expression is a quadratic with integer coefficients.

1130. The derivative of $\text{g}(x) = \tan x$ is $\text{g}'(x) = \sec^2 x$. The second Pythagorean identity states that

$$\sec^2 x \equiv 1 + \tan^2 x.$$

So, $\tan'(x) \equiv 1 + \tan^2 x$. This verifies that $\text{g}(x) = \tan x$ satisfies the differential equation.

1131. If such an identity exists, then $(2x - 1)$ is a factor of $2x^2 + 3x + c$. So, using the factor theorem,

$$\begin{aligned}&2 \times \left(\tfrac{1}{2}\right)^2 + 3 \times \tfrac{1}{2} + c = 0 \\ \Longrightarrow\ &c = -2.\end{aligned}$$

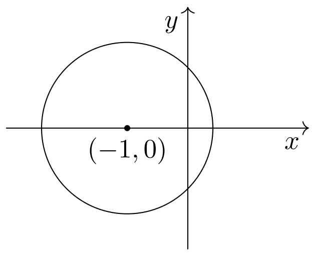
— Alternative Method —

Multiplying up, we have

$$2x^2 + 3x + c \equiv (2x - 1)(Ax + B).$$

Comparing coefficients of x^2 and x, we require that $A = 1$ and $B = 2$. This gives $c = -2$.

1132. The curve is a circle. Completing the square to $(x+1)^2 + y^2 = 2$, the centre is $(-1, 0)$.



The x axis is normal to the curve, but the y axis is not. So, the statement is false.

1133. The yearly scale factor is 1.02, so we require that $1.02^n \geq 1.25$. Hence, $n \geq \log_{1.02} 1.25 = 11.268$. Since the money is paid in as a lump sum, the return will exceed 25% after 12 years.

1134. Substituting $x = \frac{u-4}{3}$ gives

$$\begin{aligned}&\left(\tfrac{u-4}{3}\right)^2 + 6\left(\tfrac{u-4}{3}\right) + 1 \\ \equiv\ &\tfrac{1}{9}(u^2 - 8u + 16) + 2(u - 4) + 1 \\ \equiv\ &\tfrac{1}{9}u^2 + \tfrac{10}{9}u - \tfrac{47}{9}.\end{aligned}$$

1135. Irrespective of the scenario, these forces are always at right angles. In mechanics, "reaction" is the word for a force acting perpendicular to a surface; "friction" is the word for a force acting parallel to a surface. By definition, these are at right angles to one another.

1136. Since a and b are rational, $a = \frac{p}{q}$ and $b = \frac{r}{s}$, for $p, q, r, s \in \mathbb{Z}$, where $q, s \neq 0$. Subtracting and putting over a common denominator gives

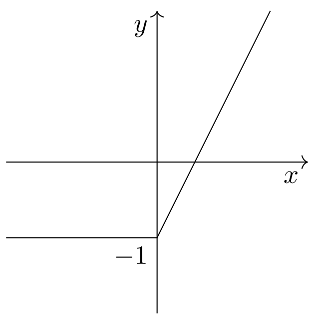
$$a - b = \frac{ps - rq}{qs}.$$

Both $ps - rq$ and qs are integers, and $qs \neq 0$, so $a - b \in \mathbb{Q}$. □

1137. (a) For negative x, $\mathrm{g}(x) = x + (-x) - 1 = -1$.

(b) For positive x, $\mathrm{g}(x) = x + x - 1 = 2x - 1$.

(c) Sketching the two straight line parts, we get



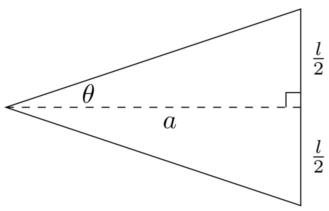
1138. Assume, for a contradiction, that f is a polynomial of degree n. Then $\mathrm{f}(x) = ax^n + ...$, where $a \neq 0$. Integrating gives

$$\int \mathrm{f}(x)\, dx = \frac{a}{n+1}x^{n+1} + \ldots.$$

This is a polynomial of degree $n+1$. But $k\,\mathrm{f}(x)+c$ has degree n. This is a contradiction. Hence, f is not a polynomial function. QED.

1139. The constants a and c are not relevant, as they translate the line segment. For unit change in t, the length is given by $\sqrt{b^2 + d^2}$. Hence, the line segment has length 1.

1140. A sector of a regular n-gon must be isosceles. We split this into two right-angled triangles:



The angle subtended by the sector is $\frac{360°}{n}$, so $\theta = \frac{180°}{n}$. The definition of the tan function gives

$$\begin{aligned} \tan \tfrac{180°}{n} &= \frac{l}{2a} \\ \Longrightarrow a &= \frac{l}{2\tan \frac{180°}{n}}. \end{aligned}$$

Using $\cot x$ as the reciprocal of $\tan x$, we reach the required result:

$$a = \tfrac{1}{2}l \cot \tfrac{180°}{n}.$$

— Nota Bene —

The reciprocal trigonometric functions are easily remembered by their third letters:

Function	Notation	Definition
Cosecant	$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$
Secant	$\sec\theta$	$\frac{1}{\cos\theta}$
Cotangent	$\cot\theta$	$\frac{1}{\tan\theta}$

1141. These APs have ordinal formulae $a_n = a + (n-1)d$ and $b_n = b + (n-1)e$, where a, b, d, e are constants. Their mean is

$$c_n = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(n-1)(d+e).$$

This is an AP with first term $\frac{1}{2}(a+b)$ and common difference $\frac{1}{2}(d+e)$. □

1142. The x coordinate of the vertex of the second parabola is $x = 2$. Reflecting this in $x = 3$ gives the x coordinate of the vertex of the first parabola: $x = 4$. Since the roots of the first parabola, which are $x = -1$ and $x = a$, must be symmetrical around the vertex, we know that $a = 9$. This gives the y coordinate of both vertices as $b = -25$.

1143. (a) Using the change of base formula,

$$\log_{a^2} b \equiv \frac{\log_a b}{\log_a a^2} \equiv \tfrac{1}{2}\log_a b.$$

(b) Expanding the notation gives

$$\begin{aligned} &\log_4 y - \log_2 y = 3 \\ \Longrightarrow\ &\tfrac{1}{2}\log_2 y - \log_2 y = 3 \\ \Longrightarrow\ &\log_2 y = -6 \\ \Longrightarrow\ &y = \tfrac{1}{64}. \end{aligned}$$

1144. The displacement over the 10 seconds is 100 m. So, the average speed is 10 ms^{-1}. Differentiating, the instantaneous speed is $v = 2t$. So, we require $2t = 10$, which gives $t = 5$.

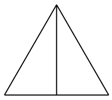
1145. The shortest distance between circles is normal to both, so it lies along the line of centres. These are $(3, -2)$ and $(11, -8)$, and the distance between them is $\sqrt{8^2 + 6^2} = 10$. Subtracting the radii, the distance is $10 - 4 - 3 = 3$ units.

1146. Setting the derivatives to zero, $-2x + k = 0$ and $-6x - 6 = 0$. Hence, the vertices of the parabolae $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ are

$$\left(\tfrac{1}{2}k, \tfrac{1}{4}k^2 + 4\right) \text{ and } (-1, 8).$$

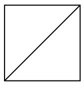
It is the y coordinate of the vertex of a (negative) parabola that defines the range, so $\frac{1}{4}k^2 + 4 = 8$. This has roots $k = \pm 4$.

1147. (a) Splitting an equilateral triangle of side length 2 in half gives two right-angled triangles with sides $(2, 1, \sqrt{3})$.



The interior angle of an equilateral triangle is $\frac{\pi}{3}$ radians, which gives $\frac{\pi}{6}$ when halved. Hence, $\cos\frac{\pi}{6} = \sqrt{3}/2$, as required.

(b) By Pythagoras, the diagonal of a unit square has length $\sqrt{2}$:



This gives $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

1148. A definite integral sums the integrand over the given domain. In this case,

$$\int_0^{12} x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^{12} = 576.$$

Dividing by the width of the domain $[0, 12]$ gives $\frac{576}{12} = 48$, as required.

1149. Differentiating twice,

$$\begin{aligned} & y = x^{\frac{1}{2}} \\ \implies & \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}} \\ \implies & \frac{d^2y}{dx^2} = -\tfrac{1}{4}x^{-\frac{3}{2}}. \end{aligned}$$

Substituting into the DE,

$$\begin{aligned} & 4y^3\frac{d^2y}{dx^2} + 1 \\ = {} & 4(\sqrt{x})^3 \cdot -\tfrac{1}{4}x^{-\frac{3}{2}} + 1 \\ \equiv {} & -x^{\frac{3}{2}} \cdot x^{-\frac{3}{2}} + 1 \\ \equiv {} & -1 + 1 \\ \equiv {} & 0, \text{ as required.} \end{aligned}$$

1150. Rewriting $a = 2^x$ as an exponential with base 3,

$$a = \left(3^{\log_3 2}\right)^x \equiv (3^x)^{\log_3 2} = b^{\log_3 2}.$$

1151. Equating gradients,

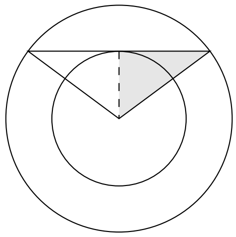
$$\begin{aligned} & \frac{-1}{-a} = \frac{6-a}{8} \\ \implies & 8 = a(6-a) \\ \implies & (a-2)(a-4) = 0 \\ \implies & a = 2, 4. \end{aligned}$$

1152. A straight is more likely without replacement. All the cards in a straight have different values. So, if a card is removed from the possibility space, then the next card is more likely to have a different value. Hence, a straight becomes more likely.

1153. Taking out $\sqrt{3}$ and simplifying $\sqrt{27}$,

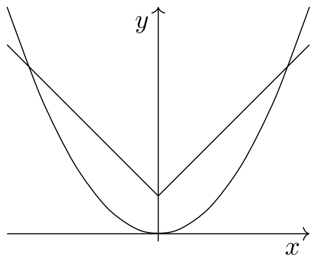
$$\begin{aligned} & \sqrt{3}x^2 - \sqrt{48}x - \sqrt{27} \\ \equiv {} & \sqrt{3}(x-2)^2 - 4\sqrt{3} - 3\sqrt{3} \\ \equiv {} & \sqrt{3}(x-2)^2 - 7\sqrt{3}. \end{aligned}$$

1154. Adding radii:



Each small triangle (e.g. the one shaded) is right-angled, with sides 15 cm, 9 cm and hence 12 cm. So, the chord has length 24 cm.

1155. The graphs are



Solving $3x^2 = x + 2$ gives intersections at $x = 1$ and therefore $x = -1$. The area is then

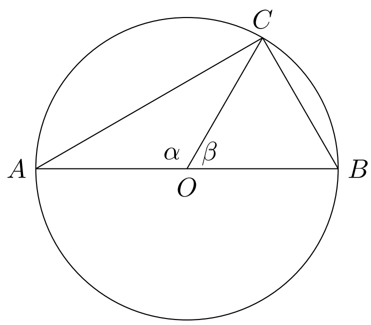
$$\begin{aligned} A &= 2\int_0^1 x + 2 - 3x^2\,dx \\ &= 2\left[\tfrac{1}{2}x^2 + 2x - x^3\right]_0^1 \\ &= 2 \cdot \tfrac{3}{2} \\ &= 3. \end{aligned}$$

1156. (a) $f'(x) = 4x^3 - 2x$, $f''(x) = 12x^2 - 2$.

(b) Solving $12x^2 - 2 = 0$ gives $x = \pm\frac{1}{\sqrt{6}}$.

(c) Since each of these values is a *single* **root** of the quadratic $12x^2 - 2 = 0$, $f''(x)$ is both **zero** and *changes sign*. Hence, there are points of inflection at $x = \pm\frac{1}{\sqrt{6}}$.

1157. Drawing in the radius to the proposed right angle and labelling the subtended angles α and β, we have



Since triangles AOC and BOC are isosceles, their other angles are $90° - \frac{1}{2}\alpha$ and $90° - \frac{1}{2}\beta$. The sum of these is $180° - \frac{1}{2}(\alpha+\beta)$, which, since α and β lie on a straight line, is $90°$. QED.

1158. (a) These are not independent. Instead, they are mutually exclusive, which is as dependent as you can get: one rules out the other.

(b) These are independent.

1159. Using ${}^n\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$,

$$\begin{aligned}\frac{n!}{3!(n-3)!} + \frac{n!}{2!(n-2)!} &= 0\\ \implies \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2} &= 0.\end{aligned}$$

Taking out a factor of $n(n-1)$,

$$\begin{aligned}n(n-1)\left(\frac{n-2}{6} - \frac{1}{2}\right) &= 0\\ \implies n(n-1)(n-5) &= 0\\ \implies n &= 0, 1, 5.\end{aligned}$$

Since the equation is only defined for $n \geq 3$, the solution is $x = 5$.

1160. (a) The position vector of P, situated a fraction λ of the way from A to B, is

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})\\ &= (1-\lambda)\mathbf{a} + \lambda\mathbf{b}.\end{aligned}$$

(b) $\overrightarrow{OQ} = \mu\mathbf{c} = \mu(\mathbf{a}+\mathbf{b})$.

(c) Setting the parameters λ and μ both to $\frac{1}{2}$, the midpoints of the diagonals of parallelogram $OACB$ have position vectors $\mathbf{p} = \mathbf{q} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$. Since the midpoints coincide, the diagonals of $OACB$ bisect each other. □

1161. Using the chain rule,

$$\begin{aligned}y &= \sqrt{8x+7}\\ \implies \frac{dy}{dx} &= \tfrac{1}{2}(8x+7)^{-\frac{1}{2}} \cdot 8\\ &= \frac{4}{\sqrt{8x+7}}.\end{aligned}$$

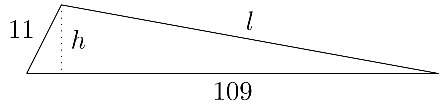
— Nota Bene —

The chain rule underpins a great many problems in mathematics. A good understanding of it goes a very long way. It boils down to knowing why the factor of 8 emerges in the above. The logic is as follows.

Replacement of x by $8x$ stretches the curve $y = \sqrt{x}$ by scale factor $\frac{1}{8}$ in the x direction. This scales gradients by 8. Translation by $-7\mathbf{i}$, however, does not affect gradients.

1162. (a) $(11, 60, 61)$ is a Pythagorean triple. So, the area of T_1 is $\frac{1}{2} \cdot 11 \cdot 60 = 330$.

(b) Not to scale, triangle T_2 is



i. The perpendicular splits the base of length 109 into two parts. Pythagoras on the left-hand triangle gives the first part as $\sqrt{11^2 - h^2}$. Subtracting this from 109, the other part is $109 - \sqrt{11^2 - h^2}$.

ii. Using the area from part (a),

$$\begin{aligned}\tfrac{1}{2} \cdot 109 \cdot h &= 330\\ \implies h &= \tfrac{660}{109}.\end{aligned}$$

iii. Pythagoras gives

$$\begin{aligned}l^2 &= \left(109 - \sqrt{11^2 - \tfrac{660}{109}^2}\right)^2 + \left(\tfrac{660}{109}\right)^2\\ &= 10000.\end{aligned}$$

So, $l = 100$.

1163. Factorising the denominator as $(b+x)^2$ gives a (double) vertical asymptote at $x = -b$. Also, rewriting as

$$y = \frac{\frac{a^2}{x^2} + \frac{2a}{x} + 1}{\frac{b^2}{x^2} + \frac{2b}{x} + 1},$$

we can see that, as $x \to \pm\infty$, $y \to 1$. Hence, $y = 1$ is a horizontal asymptote.

1164. Since n is large, the number of occurrences will be approximately proportional to probability. In the possibility space, there are 36 outcomes, of which 6 sum to seven and 3 sum to nine.

	1	2	3	4	5	6
1						7
2					7	
3				7		
4			7			9
5		7			9	
6	7			9		

Hence, with n large, a sum of seven is expected to occur around twice as often as nine.

1165. The graph has odd degree, so the curve must cross (not merely intersect) the x axis an odd number of times. This graph has exactly two roots, so these cannot both be single roots: at least one must be a repeated root. At a repeated root, the gradient is zero. □

——— Alternative Method ———

Assume, for a contradiction, that the gradient is not zero at either of the roots. Then the curve $y = \mathrm{f}(x)$ crosses the x axis exactly twice. So, for large negative x and large positive x, the sign of y is the same. Since $y = \mathrm{f}(x)$ has odd degree, this is a contradiction. So, the gradient must be zero at at least one of the roots. □

1166. (a) For every one of n routes out, there are n routes back, so the total is n^2.
(b) For every one of n routes out, there are $n-1$ routes back, so the total is $n(n-1)$.

1167. Consider the reciprocal of the gradient of L_2:

$$\begin{aligned}&\frac{1}{-(2+\sqrt{3})}\\ &= \frac{2-\sqrt{3}}{-(2+\sqrt{3})(2-\sqrt{3})}\\ &= \frac{2-\sqrt{3}}{-1}\\ &= \sqrt{3}-2.\end{aligned}$$

This is the gradient of L_3, which tells us that L_3 is the reflection of L_2 in $y = x$, i.e. in L_1. Hence, the angle between L_1 and L_2 is the same as the angle between L_1 and L_3.

1168. Taking the cube root of both sides, $\sqrt{x} + x = 1$. This is a quadratic in $\sqrt{x}$.

$$\begin{aligned}x + \sqrt{x} - 1 &= 0\\ \implies \sqrt{x} &= \frac{-1 \pm \sqrt{5}}{2}.\end{aligned}$$

The negative root doesn't produce a real value for x, since $\sqrt{x} \geq 0$ for all x. Taking the positive root,

$$\begin{aligned}\sqrt{x} &= \frac{-1+\sqrt{5}}{2}\\ \implies x &= \frac{3-\sqrt{5}}{2}.\end{aligned}$$

1169. This statement is true. Let the side lengths be x and y, the perimeter p and the area a:

$$\begin{aligned}p &= 2x + 2y\\ a &= xy.\end{aligned}$$

Substituting,

$$\begin{aligned}p &= 2x + 2\left(\tfrac{a}{x}\right)\\ \implies 2x^2 - px + 2a &= 0.\end{aligned}$$

This is a quadratic in x with a maximum of two real roots. So, there are at most two values that x can take, giving (x_1, y_1) and (x_2, y_2). But, since the original equations are symmetrical in x and y, these two must, in fact, be (x_1, y_1) and (y_1, x_1). The rectangles produced have the same set of sides, and are therefore congruent. QED.

1170. (a) Velocity is constant, so, by NII, the resultant vertical force is zero. The two vertical forces must be equal in magnitude.
(b) These are equal in magnitude by definition: they are a Newton III pair.
(c) Velocity is constant, so, by NII, the resultant horizontal force is zero. The two horizontal forces must be equal in magnitude.

1171. To find stationary points, we set

$$\frac{dy}{dx} = 3x^2 - 1 = 0.$$

This gives the SPs as

$$\left(\pm\tfrac{1}{\sqrt{3}}, 2 \mp \tfrac{2}{\sqrt{3}}\right).$$

To locate the point of inflection, we set

$$\frac{d^2y}{dx^2} = 6x = 0.$$

So, the point of inflection is at $(0, 2)$. The SPs are symmetrically located around the point $(0, 2)$, which means the three are collinear.

——— Nota Bene ———

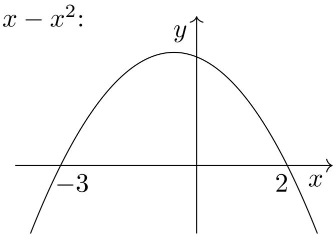
This is a general result, in fact. Every cubic has a point of inflection, and is rotationally symmetrical around this point. The stationary points, if they exist, must be images of one another under this rotation, so the three must always be collinear.

1172. The implication is backwards (or upwards on the page), with symbol $\Longleftarrow$. The implication in the other direction doesn't hold: the counterexample is, with a, b, c all distinct, $x = c$.

1173. We can place one vertex without loss of generality. Of the remaining five vertices, four are connected to the original one by an edge. Hence, the required probability is $4/5$.

1174. The equation for intersections is $\mathrm{f}(x) = \mathrm{g}(x)$. This can have degree at most two. And it must have a double root, since we know that the parabolae are tangent at a point. Since it has a double root, it can't have any others. Hence, the parabolae can't intersect elsewhere. □

1175. (a) $y = 6 - x - x^2$:



(b) The domain $[-2, 1]$ lies between the roots, which means $6 - x - x^2 > 0$ for all values of x. Hence, both square rooting and division are well defined.

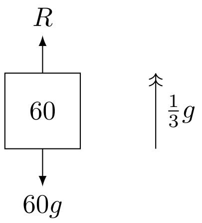
1176. Multiplying out with the binomial expansion,

$$\begin{aligned} x - (x^{\frac{3}{2}} - 3x + 3x^{\frac{1}{2}} - 1) &= 1 \\ \implies -x^{\frac{3}{2}} + 4x - 3x^{\frac{1}{2}} &= 0. \end{aligned}$$

This is a cubic in $x^{\frac{1}{2}}$:

$$\begin{aligned} & x^{\frac{1}{2}}(x - 4x^{\frac{1}{2}} + 3) = 0 \\ \implies & x^{\frac{1}{2}}(x^{\frac{1}{2}} - 3)(x^{\frac{1}{2}} - 1) = 0 \\ \implies & x^{\frac{1}{2}} = 0, 1, 3 \\ \implies & x = 0, 1, 9. \end{aligned}$$

1177. The forces on the eccentric individual are



The reading on the scales is a measurement of the reaction force exerted, converted to mass. $F = ma$ gives $R - 60g = 60 \cdot \frac{1}{3}g$, therefore $R = 80g$. Hence, the scales read 80 kg.

1178. Since the indefinite integral of g is cubic, g, as the derivative of a cubic function, must be quadratic. Hence, $y = \text{g}(x)$ is a parabola. Every parabola has precisely one stationary point, the vertex. QED.

1179. (a) This is false: $y = 0$ is a counterexample.

(b) This is true, since neither y^{-1} nor z^{-1}, being reciprocals, can equal zero.

1180. Call the shorter arc length l. Its radius length is then $r = 2l/\pi$. Hence, the radius of the longer arc is

$$R = \left(1 + \tfrac{2l}{\pi}\right).$$

So, the longer arc has length

$$\begin{aligned} & \tfrac{1}{2}\pi\left(1 + \tfrac{2l}{\pi}\right) \\ \equiv\ & \tfrac{1}{2}\pi + l. \end{aligned}$$

The perimeter gives us the following equation:

$$3l + \tfrac{1}{2}\pi + l = 4 + \tfrac{1}{2}\pi.$$

So, $l = 1$. Hence, the area of the shaded region is

$$\begin{aligned} A &= \tfrac{1}{4}\pi\left(R^2 - r^2\right) \\ &= \tfrac{1}{4}\pi\left(\left(1 + \tfrac{2}{\pi}\right)^2 - \tfrac{2}{\pi}^2\right) \\ &= \tfrac{1}{4}\pi\left(1 + \tfrac{4}{\pi}\right) \\ &= \tfrac{1}{4}\pi + 1 \end{aligned}$$

1181. The limit is well defined, as, before taking the limit, we can divide by (the non-zero) h. After this step, we can safely take the limit, giving $2x$.

The evaluation is not well defined, however, as evaluating at $h = 0$ gives $\frac{0}{0}$.

1182. (a) We rearrange as follows:

$$\begin{aligned} & a_1b_1 + a_2b_2 = 0 \\ \implies & a_1b_1 = -a_2b_2 \\ \implies & \frac{a_1}{a_2} = -\frac{b_2}{b_1}. \end{aligned}$$

So, the gradients are negative reciprocal, telling us that **a** and **b** are perpendicular.

(b) The vector **a** − **b** is the vector from B to A. And we know that the perpendicular vectors OA and OB have unit length. So, Pythagoras gives $|\mathbf{a} - \mathbf{b}| = \sqrt{2}$, as required.

1183. (a) This is true. If an object has negligible mass, then NII is $F = ma \approx 0 \cdot a = 0$. This means that resultant force must be negligible, barring astronomical accelerations (where the system breaks down anyway).

(b) This is false. A massive object such as a spacecraft is effectively weightless in deep space, yet it can still have a resultant force applied to it (by the exhaust gases ejected from thrusters, or suchlike).

1184. A counterexample is $\text{f}(x) = x$ and $\text{g}(x) = -x$.

In both cases, the composition of the function with itself (application of the function twice) gives $\text{f}^2(x) = \text{g}^2(x) = x$, but the functions themselves are distinct.

1185. (a) By Pythagoras, the side length is 10.

(b) The area of an equilateral triangle of side length l is

$$A_\triangle = \tfrac{\sqrt{3}}{4}l^2.$$

The hexagon consists of six such triangles with side length 10, so it has area

$$A_{\text{hex}} = 6 \cdot \tfrac{\sqrt{3}}{4} \times 100 = 150\sqrt{3}, \text{ as required.}$$

1186. (a) This is true. We know that $0 \le \text{f}(x) \le 1$; scaling this by positive k gives $0 \le k\,\text{f}(x) \le k$.

(b) This is not true. We don't that $\text{f}(x)$ takes *every* value in the interval $[0, 1]$, only that its outputs lie in that interval. Hence, we don't know that $k\,\text{f}(x)$ takes *every* value in $[0, k]$, which would be required for this statement about the range. The most we can say is the statement in (a).

1187. We set up the limit as follows:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

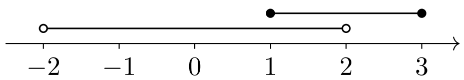
To simplify, we multiply top and bottom by the conjugate of the top. This gives

$$\begin{aligned} &\lim_{h\to 0} \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ \equiv\ &\lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ \equiv\ &\lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}. \end{aligned}$$

At this point we can cancel a factor of h, and then take the limit:

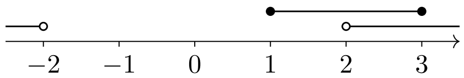
$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &\equiv \frac{1}{2\sqrt{x}}. \end{aligned}$$

1188. (a) The sets are



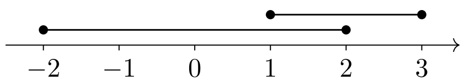
So, the intersection is $[1, 2)$.

(b) The sets are



So, the intersection is $(2, 3]$.

(c) The sets are



So, the intersection is $[1, 2]$.

1189. Both parabolae are monic. Completing the square, their equations are

$$\begin{aligned} y &= \left(x+\tfrac{3}{2}\right)^2 - \tfrac{5}{4}, \\ y &= \left(x-\tfrac{3}{2}\right)^2 - \tfrac{5}{4}. \end{aligned}$$

The vertices are at $\left(\mp\frac{3}{2}, -\frac{5}{4}\right)$. The transformation may be described as

(a) reflection in the y axis,

(b) translation by vector $3\mathbf{i}$.

1190. (a) Vertical $F = ma$ is $31.6 - 2g = 2a$, so $a = 6$ ms^{-2}. Over 4 seconds, starting from rest, this gives 24 ms^{-1}.

(b) The height attained during the first phase is $s = \frac{1}{2}\cdot 6\cdot 4^2 = 48$. In the subsequent projectile motion, the vertical displacement is given by $0 = 24^2 - 2gh$, so $h = 29.3877...$. The greatest height is $48 + 29.3877... = 77.4$ m (3sf).

1191. The large unshaded isosceles triangle has sides of length $\frac{\sqrt{2}}{2}$, hence area

$$\tfrac{1}{2}\left(\tfrac{\sqrt{2}}{2}\right)^2 = \tfrac{1}{4}.$$

The small unshaded isosceles triangles have sides of length $1 - \frac{\sqrt{2}}{2}$, hence area

$$\tfrac{1}{2}\left(1-\tfrac{\sqrt{2}}{2}\right)^2 = \tfrac{3-2\sqrt{2}}{4}.$$

The shaded area is then calculated by subtracting these areas from 1, giving

$$\begin{aligned} &1 - \tfrac{1}{4} - 2\cdot\tfrac{3-2\sqrt{2}}{4} \\ =\ &\sqrt{2} - \tfrac{3}{4}. \end{aligned}$$

1192. (a) To get zero, we require $2x = a$, so $x = \frac{1}{2}a$.

(b) One of $\mathrm{f}(x-1)$ or $\mathrm{f}(x+1)$ must be zero. So, either $x-1 = a$ or $x+1 = a$. Hence, $x = a \pm 1$.

1193. A counterexample is π and $\pi + 1$, both of which are irrational, but whose difference is 1, which is rational.

1194. Doubling and rearranging the first equation gives

$$6x - 8y = 26.$$

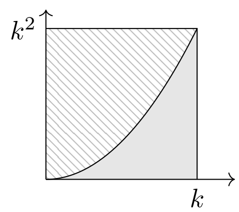
Considered as a linear graph, this is parallel to, and distinct from, the first equation. It is correct, therefore, that the standard techniques should have broken down, as there are no (x, y) points of intersection.

1195. Evaluating the integrals:

$$\begin{aligned} &\int_0^k y\,dx + \int_0^{k^2} x\,dy \\ \equiv\ &\int_0^k x^2\,dx + \int_0^{k^2} y^{\frac{1}{2}}\,dy \\ \equiv\ &\left[\tfrac{1}{3}x^3\right]_0^k + \left[\tfrac{2}{3}y^{\frac{3}{2}}\right]_0^{k^2} \\ \equiv\ &\tfrac{1}{3}k^3 + \tfrac{2}{3}k^3 \\ \equiv\ &k^3, \text{ as required.} \end{aligned}$$

— Alternative Method —

Each integral represents the area of a region. The regions are bounded by $y = x^2$, the coordinate axes and the lines $x = k$ and $y = k^2$.



Together, the regions form a rectangle with sides k and k^2. This gives a total area of k^3, as required.

1196. The quadratic formula gives the difference between the roots as $\sqrt{p^2 - 4q} = 4$ and the sum as $-p = 2$. Hence, $p = -2$, and $q = -3$.

——— Alternative Method ———

Let the roots be α and β, with $\alpha > \beta$:

$$\begin{aligned} \alpha - \beta &= 4, \\ \alpha + \beta &= 2. \end{aligned}$$

Solving these, $\alpha = 3$, $\beta = -1$. So, the quadratic is

$$\begin{aligned} &(x-3)(x+1) = 0 \\ \Longrightarrow\ &x^2 - 2x - 3 = 0. \end{aligned}$$

So, $p = -2$ and $q = -3$.

1197. (a) Equating coefficients of $\mathbf{a}$ and $\mathbf{b}$, $p+q = 4$ and $p - q = 6$. Solving simultaneously gives $p = 5$, $q = -1$.

(b) i. If the vectors were parallel, we wouldn't have been able to split the equation up into an $\mathbf{a}$ equation and a $\mathbf{b}$ equation:

$$\begin{aligned} p\mathbf{a} + q\mathbf{a} &= 4\mathbf{a} \\ p\mathbf{b} - q\mathbf{b} &= 6\mathbf{b} \end{aligned}$$

ii. If either of the vectors had been equal to zero, we wouldn't have been able to extract the coefficients from the relevant equation. For example, if $\mathbf{a}$ could be zero, then

$$\begin{aligned} &p\mathbf{a} + q\mathbf{a} = 4\mathbf{a} \\ \not\Longrightarrow\ &p + q = 4 \end{aligned}$$

1198. (a) The number of steps required to get to the nth term is $(n-1)$. Each of these steps adds the common difference d. Hence, the nth term is

$$u_n = a + (n-1)d.$$

(b) The mean of an AP is the mean of the first and last terms. Using the nth term formula, this is

$$\begin{aligned} &\tfrac{1}{2}\big(a + (a + (n-1)d)\big) \\ \equiv\ &\tfrac{1}{2}(2a + (n-1)d). \end{aligned}$$

For the partial sum formula, we multiply the mean by the number of terms in the sequence:

$$S_n = \tfrac{1}{2}n(2a + (n-1)d).$$

1199. The expression is a gradient formula for $y = \mathrm{h}(x)$: the numerator is Δy, and the denominator is Δx. Since h is a linear function, $y = \mathrm{h}(x)$ is a straight line, which means this gradient is constant. □

1200. Multiplying up gives

$$\begin{aligned} 2axy + y^2 &= 4y, \\ 6x^2 - 2axy &= 2x \end{aligned}$$

Adding the two equations,

$$\begin{aligned} &6x^2 + y^2 = 2x + 4y \\ \Longrightarrow\ &6x^2 - 2x = 4y - y^2. \end{aligned}$$

——— End of 12th Hundred ———